Hydrodynamic limit for a Facilitated Exclusion Process with open boundaries

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The model

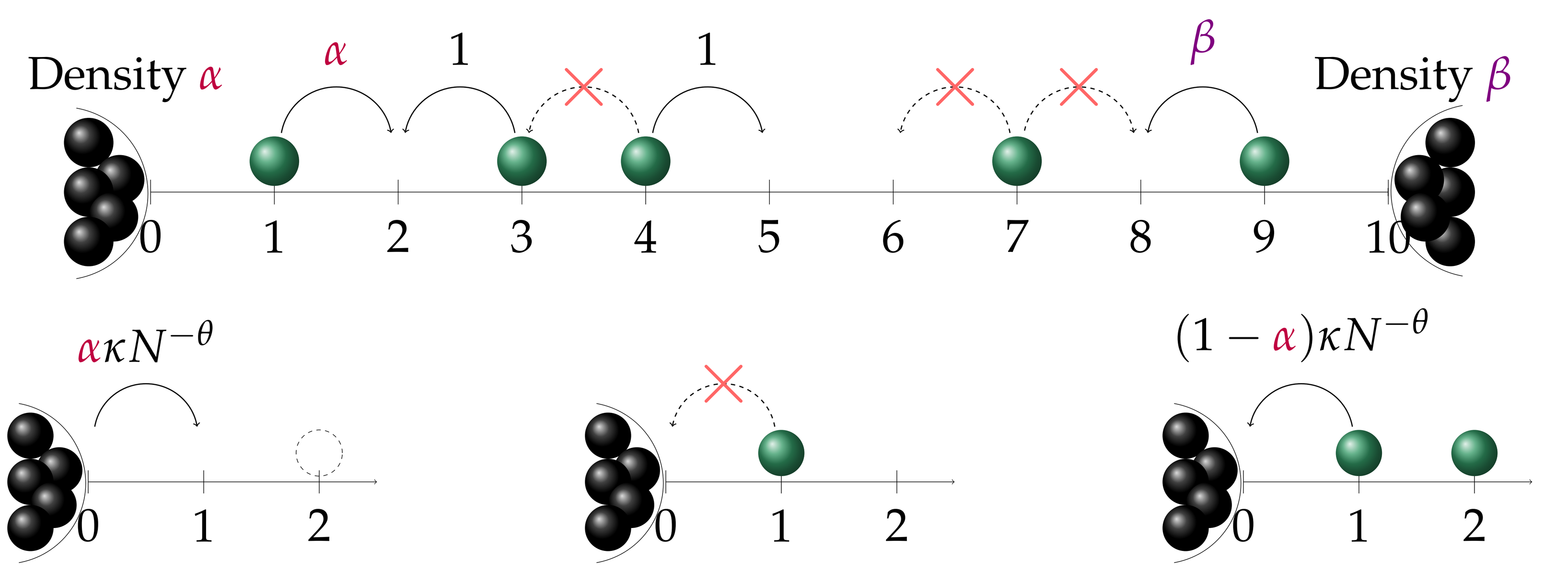


Figure 1: Illustration (for $N = 10$) of the bulk dynamics (above) and of the boundary exchanges with left reservoir (below).

Bulk dynamics

For $N \in \mathbb{N}^*$, consider the lattice $\Lambda_N = \{1, \dots, N-1\}$. On each site of this lattice, put 1 or 0 particles to get a *configuration* $\eta = (\eta_x)_{x\in\Lambda_N}$ in $\Omega_N = \{0,1\}^{\Lambda_N}$.
Then, particles jump at rate 1 to each neighbouring site provided the target site is empty (*exclusion rule*) and the other neighbouring site is occupied (*kinetic constraint*). This dynamics is described by the infinitesimal generator

$$\mathscr{L}_0 f(\eta) = \sum_{x=1}^{N-2} c_{x,x+1}(\eta)\left[f(\eta^{x,x+1}) - f(\eta)\right]$$

where $\qquad c_{x,x+1}(\eta) = \eta_{x-1}\eta_x(1-\eta_{x+1}) + (1-\eta_x)\eta_{x+1}\eta_{x+2}. \qquad (\star)$

Boundary dynamics

At both ends, we add stochastic reservoirs able to exchange particles with the system. Let $\alpha, \beta \in (0,1)$ represent their respective density, and also $\kappa > 0$, $\theta \in \mathbb{R}$ be parameters ruling the speed of those exchanges.
In $(\star)$, we set by convention

$$\eta_0 = \alpha \qquad \text{and} \qquad \eta_N = \beta.$$

The exchange dynamics with the left reservoir is described by the generator

$$\mathscr{L}_\ell f(\eta) = \frac{\kappa}{N^\theta}\big(\alpha(1-\eta_1) + (1-\alpha)\eta_1\eta_2\big)\left[f(\eta^1) - f(\eta)\right].$$

The same happens on the right with β instead of α, giving a generator $\mathscr{L}_r$.

Phase transition at the critical density $\rho_c = \frac{1}{2}$

Particles that are isolated cannot move, we call them frozen particles. The other particles are called active. Forget one moment about boundaries, imagine a periodic system with total density ρ.

- If $\rho \le \frac{1}{2}$, then all particles will become frozen after a transience time.
- If $\rho > \frac{1}{2}$, then there will always remain active particles. After a transience time, the holes will become isolated. We end in an ergodic configuration.

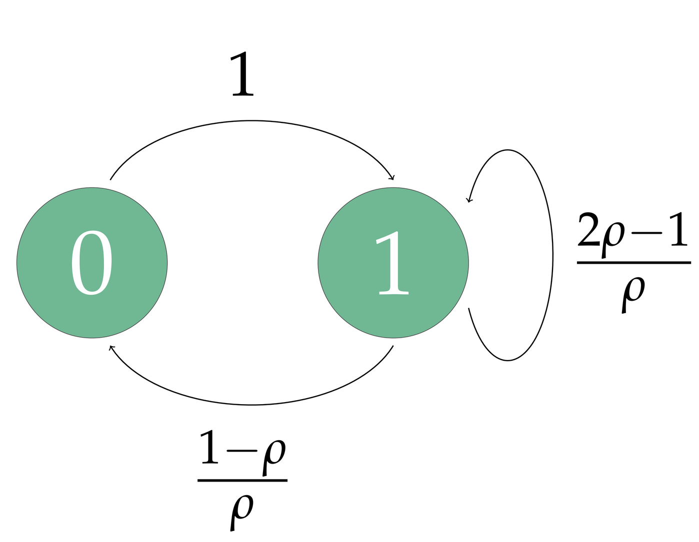
Ergodic component: $\mathscr{E}_N = \{\eta \in \Omega_N \,:\, \eta_x + \eta_{x+1} \ge 1,\ \forall x \in [\![1, N-2]\!]\}$
In the presence of reservoirs, impossible to evolve towards frozen states, so

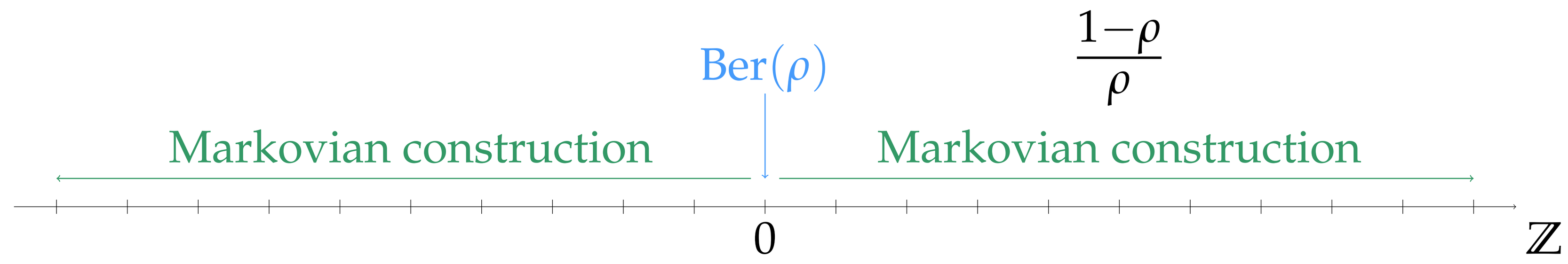
There is a unique stationary state $\bar{\mu}^N$ concentrated on $\mathscr{E}_N$.

Stationary states

Grand-canonical measures on $\mathbb{Z}$

For all $\rho > \frac{1}{2}$, there is a measure π_ρ on $\mathbb{Z}$ which is stationary, translation invariant, and concentrated on the ergodic component. (2)





$$\pi_\rho(\eta_{x+1} = 1 | \eta_x = 1) = \mathfrak{a}(\rho) \qquad \text{and} \qquad \pi_\rho(\eta_{x+1} = 1 | \eta_x = 0) = 1$$

Relation between total density and active density:

$$\mathfrak{a}(\rho) = \frac{2\rho - 1}{\rho} \iff \rho = \frac{1}{2 - \mathfrak{a}(\rho)}$$

Boundary-driven setting

Equilibrium case $\alpha = \beta$

$\bar{\mu}^N$ is the restriction to Λ_N of the measure $\pi_{\bar{\rho}(\alpha)}$, with density

$$\bar{\rho}(\alpha) = \frac{1}{2-\alpha}$$

Non-equilibrium case $\alpha \neq \beta$

Long-range correlations
$\hookrightarrow$ No explicit expression for $\bar{\mu}^N$...

One information though:
The active density is affine.

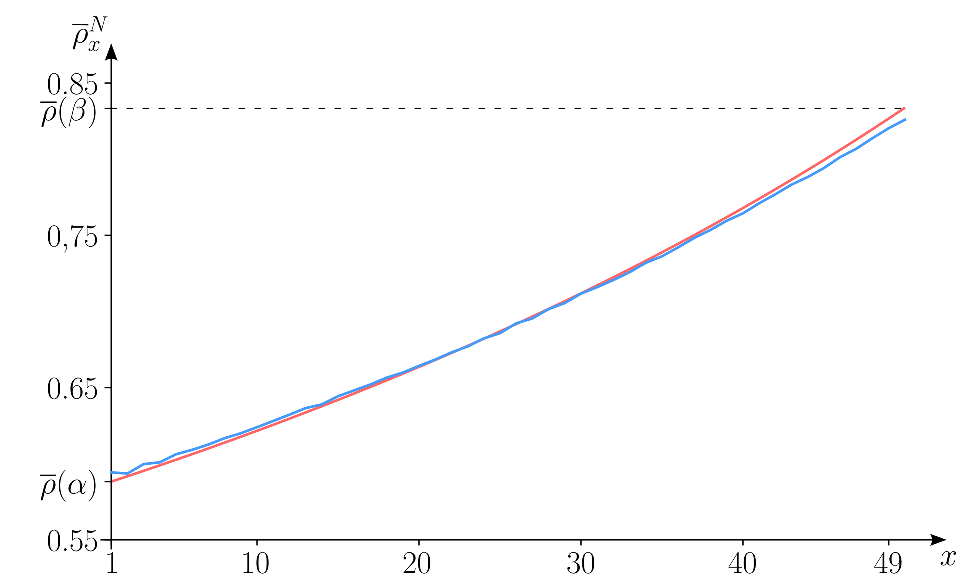


Figure 2: Simulation of the density under $\bar{\mu}^N$ (blue) and stationary profile of the hydrodynamic equation (red) in the case $\theta = 0$.

Main result: Hydrodynamic limit

Theorem [Da Cunha, Erignoux, Simon, 2024]

Consider the Markov process $(\eta(t))_{t\ge 0}$ driven by the diffusively accelerated generator $N^2(\mathscr{L}_0 + \mathscr{L}_\ell + \mathscr{L}_r)$, and starting from an initial state ν_0^N. Assume that ν_0^N is concentrated on $\mathscr{E}_N$, and that under this state, the particles are distributed according to a Lipschitz-continuous profile $\rho^{\text{ini}} : [0,1] \longrightarrow (\frac{1}{2}, 1]$. Then, for any $t \in [0,T]$, any $\delta > 0$ and any continuous function $G : [0,1] \longrightarrow \mathbb{R}$, we have

$$\lim_{N\to+\infty} \mathbb{P}_{\nu_0^N}\left(\left|\frac{1}{N}\sum_{x\in\Lambda_N} G\Big(\frac{x}{N}\Big)\eta_x(t) - \int_0^1 G(u)\rho_t(u)\mathrm{d}u\right| > \delta\right) = 0$$

where ρ is the unique weak solution to the *fast diffusion equation*

$$\begin{cases} \partial_t \rho = \partial_u^2 \mathfrak{a}(\rho) \\ \rho_0(\cdot) = \rho^{\text{ini}}(\cdot) \end{cases}$$

with, for all $t \in [0,T]$,

- if $\theta < 1$, *Dirichlet boundary conditions*

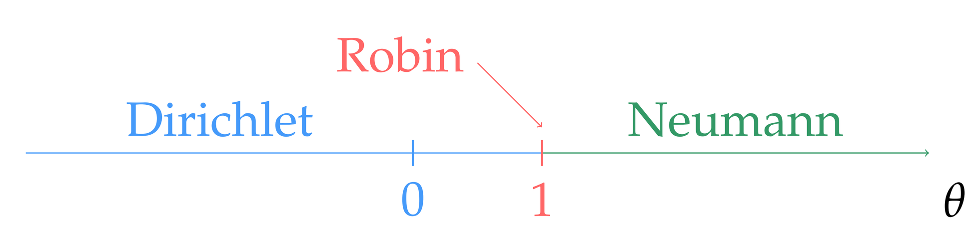
$$\rho_t(0) = \frac{1}{2-\alpha} \qquad \text{and} \qquad \rho_t(1) = \frac{1}{2-\beta}.$$

- if $\theta = 1$, *Robin boundary conditions*

$$\partial_u \mathfrak{a}(\rho_t)(0) = \kappa\big(\mathfrak{a}(\rho_t(0)) - \alpha\big) \qquad \text{and} \qquad \partial_u \mathfrak{a}(\rho_t)(1) = \kappa\big(\beta - \mathfrak{a}(\rho_t(1))\big).$$

- if $\theta > 1$, *Neumann boundary conditions*

$$\partial_u \mathfrak{a}(\rho_t)(0) = \partial_u \mathfrak{a}(\rho_t)(1) = 0.$$



Ingredients of the proof

- Build a reference measure approximating the unknown stationary state $\bar{\mu}^N$, relying on the Markov construction and the few information we have.
- Adapt Guo, Papanicolaou and Varadhan's *entropy method* to non-product stationary state, and non-equilibrium, non-translation invariant setting.

Open questions: what if...

- ...we don't start straight from the ergodic component?
  $\hookrightarrow$ We need an estimation of the transience time (without using mappings).
- ...we don't start in the supercritical phase?
  $\hookrightarrow$ We expect a Stefan problem like in (3), with the same boundary conditions.

References

[1] Da Cunha, Erignoux, and Simon, "Hydrodynamic limit for an open facilitated exclusion process with slow/fast boundaries," 2024. arXiv:2401.16535.

[2] Blondel, Erignoux, Sasada, and Simon, "Hydrodynamic limit for a facilitated exclusion process," *Annales de l'IHP - Probabilités et Statistiques*, 2020.

[3] Blondel, Erignoux, and Simon, "Stefan problem for a non-ergodic facilitated exclusion process," *Probability and Mathematical Physics*, 2021.